Mid sem solutions
Elementary Geometry (MTH312)

1. Consider the real Cartesian plane, whose set of points is the set of ordered pairs

$$\mathscr{P} = \{(x, y) \mid x, y \in \mathbb{R}\},$$

and the set of lines to be the set containing solutions of linear equations

$$\mathscr{L} = \{L \subset \mathscr{P} \mid \exists (a_L, b_L, c_L) \in S, \text{ such that } (x, y) \in L \iff a_L x + b_L y + c_L = 0\},$$

where $S \in \mathbb{R}^3$ is the set of all triples (a, b, c) such that $a \neq 0$ or $b \neq 0$ (or both).

1a. Check that the real Cartesian plane satisfies the axioms of incidence.
1b. Can you give a definition of notion of betweenness in real Cartesian plane?
1c. Prove the betweenness axioms B1, B2 and B3 for this notion of betweenness.

Proof. 1a. Let (a_1, b_1) and (a_2, b_2) be two distinct points. One can write an equation for this line : $(a_2 - b_2)x - (a_1 - b_1)y + a_1 b_2 - a_2 b_1 = 0$. This proves I1. I2 is trivially satisfied. I3 can be checked by considering $(0, 0)$, $(0, 1)$, $(1, 0)$. Line containing all these should be of the form $0 = 0$, as is seen by substitution.

1b. $(a, b) * (c, d) * (e, f)$ if and only if they line on a line and $a * c * e$ and $b * d * e$, where $r * s * t$ for three real numbers iff one of $r \leq s \leq t$ or $r \geq s \geq t$ holds.

1c. Note that by definition, a point is between two other points, only if all of them lie on a line. Also on $\mathbb{R}$, $r*s*t \iff t*s*r$. Thus $(a, b)*(c, d)*(e, f)$ iff $a*c*e$ and $b * d * f$ iff $e * c * a$ and $f * d * b$ iff $(e, f) * (c, d) * (a, b)$. This completes the proof of B1. For B2, given $A = (a, b)$ and $B = (c, d)$, consider $(2c - a, 2d - b)$. If $a \leq c$, $2c - a \geq 2c - c = c$. Therefore, $a \leq c \leq 2c - a$, or $a * c * (2c - a)$. Similarly, $a \geq c$ says that $c = 2c - c \geq 2c - a$, therefore $a \geq c \geq 2c - a$, again proving $a * c * (2c - a)$. Similarly, one concludes $b * d * (2d - b)$.

On $\mathbb{R}$, given three distinct numbers, only one of them lies between the other two. Now given (a, b), (c, d) and (e, f), all on the same line, either a, c and e are distinct or b, d and f are distinct (or both). Suppose a, c and e are distinct. Without loss of generality, assume that $a * c * e$. Suppose $a < c < e$. If the line L passing through these points have $a_L = 0$, then $b = d = f$. If $a_L > 0$, depending on the sign of b_L (which cannot be 0), either $b < d < f$ or $b > d > f$. In any case, $b * d * f$. For $a_L < 0$ we can argue similarly. Thus (c, d) would then lie between (a, b) and (e, f). This proves B3.

□

2. Consider the quadratic equation $5x^2 + 5y^2 + 6x + 8y = 0$. What kind of conic is it?

Proof. Write the equation as

$$\begin{aligned}
0 &= x^2 + y^2 + 2 \cdot \frac{3}{5}x + 2 \cdot \frac{4}{5}y \\
&= x^2 + 2 \cdot \frac{3}{5}x + \left(\frac{3}{5}\right)^2 + y^2 + 2 \cdot \frac{4}{5}y + \left(\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2 \\
&= \left(x + \frac{3}{5}\right)^2 + \left(y + \frac{4}{5}\right)^2 - 1
\end{aligned}$$

which is a circle of radius 1 with centre $(-3/5, -4/5)$. □

3. In this problem, we are on the real Cartesian plane. Suppose $R^{90^\circ}_{(0,0)}$ be the rotation around the origin, 90° in the counter-clockwise direction. Similarly let $R^{270^\circ}_{(1,1)}$ be the rotation around the point $(1,1)$, 90° in the clockwise direction. What can you say about $R^{270^\circ}_{(1,1)} \circ R^{90^\circ}_{(0,0)}$ Can you describe it as a single rigid motion, like a translation, rotation, reflection or glide reflection?

Proof. Let l be the x-axis, m be the line passing through the origin, $(0,0)$ and $(1,1)$, and n be the line $y=1$. Note that $R^{90^\circ}_{(0,0)}$ is a composition of two reflections $\mathfrak{R}_m \circ \mathfrak{R}_l$, *where* $\mathfrak{R}_L$ *denotes the reflection along the line* L. Similary, $R^{270^\circ}_{(1,1)} = \mathfrak{R}_n \circ \mathfrak{R}_m$. Therefore, $R^{270^\circ}_{(1,1)} \circ R^{90^\circ}_{(0,0)} = \mathfrak{R}_n \circ \mathfrak{R}_m \circ \mathfrak{R}_m \circ \mathfrak{R}_l = \mathfrak{R}_n \circ \mathfrak{R}_m$ is the composition of two reflections along parallel lines. The orthogonal vector sending l to m is $(0,1)$. Therefore, **the compostion is translation by the vector $(0,2)$.** □

4. Identify the Cartesian plane with the complex plane $Z = X + \sqrt{-1}Y \in \mathbb{C} \leftrightarrow (X,Y)$.

4a. Prove that the equation of a straight line passing through B and C in $\mathbb{C}$ is given by $\bar{A}Z + A\bar{Z} = \sqrt{-1}(B\bar{C} - \bar{B}C)$, where $A = \sqrt{-1}(B-C)$.

4b. Prove that the perpendicular to the line $A\bar{Z} + \bar{A}Z = c$, for $A \in \mathbb{C}$, $c \in \mathbb{R}$, at a point B lying on the line is given by

$$\overline{(\sqrt{-1}A)}Z + (\sqrt{-1}A)\bar{Z} = \overline{(\sqrt{-1}A)}B + (\sqrt{-1}A)\bar{B}.$$

4c. Using the above, or otherwise, prove that the equation of tangent at a point B on a circle $Z\bar{Z} - C\bar{Z} - \bar{C}Z + C\bar{C} = r^2$ is given by

$$(B-C)\bar{Z} + (\bar{B} - \bar{C})Z = B\bar{B} - C\bar{C} + r^2.$$

Proof. 4a. Let us denote $\sqrt{-1}$ by i. Then, a general equation of a line is $A\bar{Z} + \bar{A}Z = c$. Since this line passes through B and C, we have

$$\begin{aligned} A\bar{B} + \bar{A}B &= c \\ A\bar{C} + \bar{A}C &= c \\ \text{Therefore, } A(\bar{B} - \bar{C}) + \bar{A}(B - C) &= 0 \end{aligned}$$

Hence $A(\bar{B} - \bar{C})$ is purely imaginary. Suppose it is $\lambda' i$. Therefore,

$$\begin{aligned} A &= \frac{i\lambda'}{\bar{B} - \bar{C}} = \frac{i\lambda'}{\|B - C\|}(B - C) \\ &= i\lambda(B - C), \end{aligned}$$

for some $\lambda \in \mathbb{R}$.

Now substituting $Z = B$,

$$\begin{aligned} c = \bar{A}B + A\bar{B} &= \overline{i\lambda(B-C)}B + i\lambda(B-C)\bar{B} \\ &= -i\lambda\bar{B}B + i\lambda\bar{C}B + i\lambda B\bar{B} - i\lambda C\bar{B} \\ &= i\lambda(B\bar{C} - \bar{B}C). \end{aligned}$$

Thus the equation reduces to

$$i\lambda(B-C)\bar{Z} + \overline{i\lambda(B-C)}Z = i\lambda(B\bar{C} - \bar{B}C)$$

or equivalently, after cancelling λ

$$\{i(B-C)\}\bar{Z} + \overline{\{i(B-C)\}}Z = i(B\bar{C} - \bar{B}C).$$

This completes the proof.

4b. Note that multiplication by i corresponds to rotation by $90°$. Also the lines $A\bar{Z} + \bar{A}Z = c$ for different values of c are parallel to each other.

Now consider the line $L : A\bar{Z} + \bar{A}Z = 0$, the line parallel to the given line passing through the origin $0 \in \mathbb{C}$. The points on the perpendicular line M satisfies the condition that a $90°$ rotation on the points of M give points on L, That is, if $W \in M$, $iW \in L$. That is,

$$A\overline{(iW)} + \bar{A}(iW) = 0$$

is the equation for M. Rewriting the equation of M, and multiplying by -1, we get

$$(iA)\bar{W} + \overline{(iA)}W = 0.$$

The line we seek should therefore be of the form $(iA)\bar{W} + \overline{(iA)}W = c$ and it passes through B. Thus,

$$c = iA\bar{B} + \overline{(iA)}B$$

as was to be proved.

4c. We need to find the line perpendicular to the line BC which passes through B. The line BC has the formula

$$i(B - C)\bar{Z} + \overline{(i(B - C))}Z = i(B\bar{C} - \bar{B}C).$$

Let $A = i(B - C)$. The line perpendicular to this, passing through B has the formula,

$$(iA)\bar{Z} + \overline{(iA)}Z = (iA)\bar{B} + \overline{(iA)}B$$

which we simplify as

$$\begin{aligned}
-(B - C)\bar{Z} - (\bar{B} - \bar{C})Z &= -(B - C)\bar{B} - (\bar{B} - \bar{C})B, \\
(B - C)\bar{Z} + (\bar{B} - \bar{C})Z &= (B - C)\bar{B} + (\bar{B} - \bar{C})B, \\
(B - C)\bar{Z} + (\bar{B} - \bar{C})Z &= B\bar{B} - C\bar{B} + \bar{B}B - \bar{C}B \\
&= B\bar{B} - C\bar{C} + B\bar{B} - C\bar{B} - B\bar{C} + C\bar{C} \\
&= B\bar{B} - C\bar{C} + (B - C)\overline{(B - C)} \\
&= B\bar{B} - C\bar{C} + \|B - C\|^2 \\
&= B\bar{B} - C\bar{C} + r^2
\end{aligned}$$

as was to be proved.

□

5. Show that under circular inversion with respect to the unit circle centered at the origin, a circle with centre C and radius r, inverts into a circle with

$$\text{centre } = \frac{C}{C\bar{C} - r^2}; \qquad \text{radius} = \frac{r}{C\bar{C} - r^2}.$$

Proof. Suppose W be a point in the inverted circle. This means that its inversion, $1/\bar{W}$ lies in the original circle, that is

$$\left\| \frac{1}{\bar{W}} - C \right\| = r.$$

Squaring and expanding we get the following sequence of equations

$$\frac{1}{\bar{W}}\frac{1}{W} - \frac{C}{W} - \frac{\bar{C}}{\bar{W}} + C\bar{C} = r^2;$$
$$1 - C\bar{W} - \bar{C}W + W\bar{W}(C\bar{C} - r^2) = 0;$$
$$W\bar{W} - \frac{C}{C\bar{C}-r^2}\bar{W} - \frac{\bar{C}}{C\bar{C}-r^2}W + \frac{1}{C\bar{C}-r^2} = 0;$$

Setting $A = C/(C\bar{C} - r^2)$ the equation reduces to

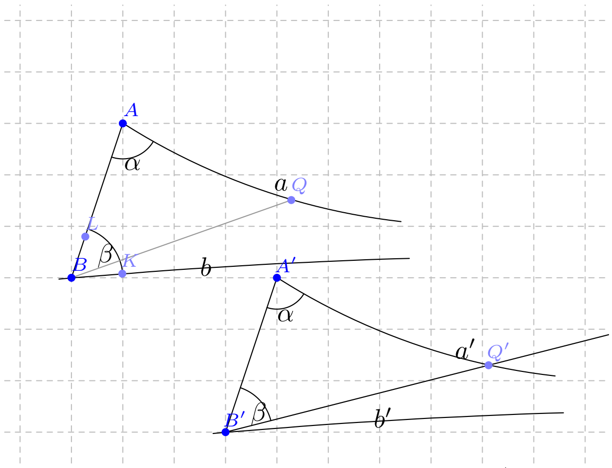
$$W\bar{W} - A\bar{W} - \bar{A}W + \frac{1}{C\bar{C}-r^2} = 0;$$
$$W\bar{W} - A\bar{W} - \bar{A}W + A\bar{A} + \frac{1}{C\bar{C}-r^2} - A\bar{A} = 0;$$
$$\begin{aligned}(W-A)\overline{(W-A)} &= A\bar{A} - \frac{1}{C\bar{C}-r^2} \\ &= \frac{C\bar{C}}{(C\bar{C}-r^2)^2} - \frac{1}{C\bar{C}-r^2} \\ &= \frac{C\bar{C} - C\bar{C} + r^2}{(C\bar{C}-r^2)^2} \\ &= \left(\frac{r}{C\bar{C}-r^2}\right)^2.\end{aligned}$$

which is nothing but a circle with

$$\text{centre} = A = \frac{C}{C\bar{C}-r^2} \quad \text{and} \quad \text{radius} = \frac{r}{C\bar{C}-r^2}.$$

□

6. Suppose we are given rays $\overrightarrow{Aa} \,|||\, \overrightarrow{Bb}$ and $\overrightarrow{A'a'} \,|||\, \overrightarrow{B'b'}$. Also assume that $AB \cong A'B'$, and $\angle BAa \cong \angle B'A'a'$. Prove then $\angle ABb \cong \angle A'B'b'$.



Proof.

Suppose that $\angle A'B'b' > \angle ABb$. Then let $\overrightarrow{B'Q'}$ be the ray such that $\angle ABb = \angle A'B'Q' = \beta$. Since this ray is in the interior of $\angle A'B'b'$, it must meet $\overrightarrow{A'a'}$. Let the point of intersection be Q'. Mark Q on $\overrightarrow{Aa}$ such that $AQ \cong A'Q'$. Join BQ.

In triangles ABQ and $A'B'Q'$, $BA \cong B'A'$ (given), $\angle BAQ \cong \angle B'A'Q'$ (given) and $AQ \cong A'Q'$ (by construction). Therefore by SAS (Axiom C6), $\triangle ABQ \cong \triangle A'B'Q'$. Thus $\angle ABQ \cong \angle A'B'Q'$. Now by construction $\angle A'B'Q' \cong \angle ABb$. Therefore, $\angle ABQ \cong \angle ABb$ which is only possible if $Q \in \overrightarrow{Bb}$. But that would imply that $\overrightarrow{Aa}$ intersects $\overrightarrow{Bb}$ which contradicts the fact that they are limiting parallels. Therefore $\angle A'B'b' \leq \angle ABb$. Now reversing the roles of the primed and the unprimed vertices, the same argument will say that $\angle ABb \leq \angle A'B'b'$, and hence they are equal. $\square$